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On the Measure of Approximation for Some Linear Means of Trigonometric Fourier Series

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gonometric Series," Vols. I, II, Pergamon Press, New York, 1964]. Our idea consists in representing φ by the orthogonal system φ_j to extend results previously known for the Rogosinski method to arbitrary approximation methods. We illustrate this by proving two asymptotic estimates for the measure of approximation. © 1997 Academic Press

1. INTRODUCTION

Let us consider the triangular λ-means

$$U_n(f, x) := \frac{A_0}{2} + \sum_{k=1}^{n} \lambda_k(n)(A_k \cos kx + B_k \sin kx) \tag{1}$$

of the real Fourier series of a 2π-periodic continuous function $f \in C_{2\pi}$. In approximation theory [2, 6] the following problem is set: find an asymptotic expansion for the quantity

$$e(A, U_n) := \sup_{f \in A} \| f - U_n f \|_{C_{2\pi}} \tag{2}$$

called the measure of approximation of the class $A \subset C_{2\pi}$ by the operator $U_n \colon C_{2\pi} \to C_{2\pi}$. This problem is often referred to as the Kolmogorov–Nikolskiĭ problem due to the introductory papers [5, 9] by these authors.

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Afterward a number of authors [7, 8, 12] became interested in this problem. For more detailed references see [11], which is entirely dedicated to the Kolmogorov–Nikolskiĭ problem. Our paper is related closely to the paper [3], where the authors solved the Kolmogorov–Nikolskiĭ problem for the Sobolev classes W^r $(r \in \mathbf{N})$ by the Rogosinski means. The class W^r will be defined as the set of all functions $f \in C_{2\pi}$ which are $(r-1)$-times differentiable everywhere, $f^{(r-1)}$ is absolutely continuous on $[-\pi, \pi]$, and $\|f^{(r)}\|_\infty \leqslant 1$.

In this paper we present a new general method for solving the Kolmogorov–Nikolskiĭ problem. A short version of our method appeared in [4]. Unfortunately, there was a misprint in Theorem 1. In the second equality $2/\pi$ must be replaced by $4/\pi^2$.

Let us describe our method more closely. The triangular λ-means (1) are called the Rogosinski means if $\lambda_k(n) = \cos(k\pi/2n)$ and were introduced by Rogosinski in [10]. These means are a special case of the means (1) defined by a continuous function $\varphi \in C_{[0,1]}$ with $\varphi(0)=1$, $\varphi(1)=0$, for which $\lambda_k(n) = \varphi(k/n)$. Therefore, the Rogosinski means are defined by $\varphi_1(t) := \cos(\pi t/2)$.

In Section 2 we are interested in the generalized Rogosinski means, denoted by $R_{n,j}$, which were introduced in the same paper [10] by Rogosinski and which may be defined by $\varphi_j(t) := \cos(j-1/2)\pi t$ for all natural $j \in \mathbf{N}$. They are not widely used in approximation theory except for the classical case $j=1$. Perhaps the reason is the fact that they have for all $j \in \mathbf{N}$ the same order of approximation as the classical one.

However, the system $\{\varphi_j\}$ $(j \in \mathbf{N})$ has a remarkable property—it is an orthogonal system on $[0, 1]$ with the boundary conditions $\varphi_j(0)=1$, $\varphi_j(1)=0$ for all $j \in \mathbf{N}$. This circumstance inspired us to consider arbitrary means (1) defined by $\varphi(k/n) = \lambda_k(n)$ for which the Fourier representation

$$\varphi = \sum_{j=1}^{\infty} a_j \varphi_j, \qquad a_j := 2\int_0^1 \varphi \cdot \varphi_j \tag{3}$$

holds. It is quite surprising that some properties of the generalized Rogosinski means $R_{n,j}$ defined by φ_j also hold for arbitrary summation methods (1) defined by φ, provided the representation (3) is valid. Indeed, if we assume that the series in (3) converges absolutely, then due to the boundary conditions $\varphi(0)=1$, $\varphi_j(0)=1$ the equality

$$\sum_{j=1}^{\infty} a_j = 1$$

holds which implies

$$f - U_n f = \sum_{j=1}^{\infty} a_j (f - R_{n,j} f). \tag{4}$$

This equality proposes that U_n has the same approximation properties as $R_{n,j}$, which are the same for each $j \in \mathbf{N}$.

In Section 3 we shall consider the quantity (2) for the operators U_n defined by φ for which (3) is valid.

2. THE GENERALIZED ROGOSINSKI MEANS

To simplify the notations let $m_j := (j-1/2)\pi$. The Rogosinski means $R_{n,j}$ defined by $\varphi_j(t) = \cos m_j t$ form for each $j \in \mathbf{N}$ a bounded linear operator on $C_{2\pi}$ into $C_{2\pi}$. More precisely, we will find an exact asymptotic expression for the norms $\|R_{n,j}\|$. For this purpose we use a result which has essentially been proved in [12]. In fact, if the means (1) are defined by a continuous function $\varphi \in C_{[0,1]}$, $\varphi(0)=1$, $\varphi(1)=0$ for which $\varphi(k/n) = \lambda_k(n)$, then

$$\sup_{n \in \mathbf{N}} \|U_n\| = \frac{2}{\pi} \int_0^\infty \left| \int_0^1 \varphi(t) \cos xt \, dt \right| dx. \tag{5}$$

THEOREM 1. *For all* $j \in \mathbf{N}$,

$$\sup_{n \in \mathbf{N}} \|R_{n,j}\| = \frac{2}{\pi} \sum_{k=0}^{2j-2} \int_0^\pi \frac{\sin t}{t + k\pi} \, dt = \frac{4}{\pi^2} \log j + O(1).$$

Proof. Let us denote

$$\psi_j(x) := \int_0^1 \varphi_j(t) \cos xt \, dt, \tag{6}$$

where $\varphi_j(t) = \cos m_j t$. Then for $x \neq m_j$ we have

$$\psi_j(x) = (-1)^{j+1} \frac{m_j \cos x}{m_j^2 - x^2} = \frac{1}{2} \left(\frac{\sin(x - m_j)}{x - m_j} + \frac{\sin(x + m_j)}{x + m_j} \right). \tag{7}$$

Hence, if $x \in (m_{l-1}, m_l)$ $(l \in \mathbf{N}, m_0 := 0)$, then the sign-function of ψ_j has the values

$$\operatorname{sgn} \psi_j(x) = \begin{cases} (-1)^{j+l}, & 1 \leqslant l \leqslant j, \\ (-1)^{j+l+1}, & l \geqslant j+1. \end{cases}$$

We split the integral (5) for φ_j into parts, and therefore

$$\int_0^\infty |\psi_j| = \sum_{l=1}^{j} \int_{m_{l-1}}^{m_l} (-1)^{j+l} \psi_j + \sum_{l=j+1}^{\infty} \int_{m_{l-1}}^{m_l} (-1)^{j+l+1} \psi_j. \tag{8}$$

If we denote, as usual, the integral sine by

$$\mathrm{Si}(x) := \int_0^x \frac{\sin t}{t}\, dt,$$

then we obtain by (7)

$$2\int_{m_{l-1}}^{m_l} \psi_j = \begin{cases} \mathrm{Si}(j\pi) - \mathrm{Si}(j-1)\pi, & l=1, \\ \mathrm{Si}(l-j)\pi - \mathrm{Si}(l-j-1)\pi & \\ \qquad + \mathrm{Si}(l+j-1)\pi - \mathrm{Si}(l+j-2)\pi, & l \geqslant 2. \end{cases} \tag{9}$$

The first sum in (8) can be written as

$$\begin{aligned} 2\sum_{l=1}^{j} \int_{m_{l-1}}^{m_l} (-1)^{j+l}\psi_j &= (-1)^j \Bigg[\mathrm{Si}(j-1)\,\pi - Si(j\pi) + \sum_{l=2}^{j} (-1)^l (\mathrm{Si}(l-j)\pi \\ &\qquad - \mathrm{Si}(l-j-1)\pi + \mathrm{Si}(l+j-1)\pi - \mathrm{Si}(l+j-2)\pi) \Bigg] \\ &= \mathrm{Si}(2j-1)\pi + 2\sum_{l=1}^{2j-2} (-1)^{l+1}\,\mathrm{Si}(l\pi). \end{aligned} \tag{10}$$

For the second sum in (8) we have to prove that the series converges. Let $n > j+1$, then the partial sum of the series in (8) can be computed in view of (9) which gives

$$\begin{aligned} 2\sum_{l=j+1}^{n} \int_{m_{l-1}}^{m_l} (-1)^{j+l+1}\psi_j &= \sum_{l=j+1}^{n} (-1)^{j+l+1} (\mathrm{Si}(l-j)\,\pi - \mathrm{Si}(l-j-1)\pi \\ &\qquad + \mathrm{Si}(l+j-1)\pi - \mathrm{Si}(l+j-2)\pi) \\ &= (-1)^{n+j+1} (\mathrm{Si}(n-j)\,\pi - \mathrm{Si}(n+j-1)\pi) \\ &\qquad + \mathrm{Si}(2j-1)\pi + 2\sum_{l=2j-1}^{n+j-1} (-1)^l\,\mathrm{Si}(l\pi) \\ &\qquad - 2\sum_{l=1}^{n-j-1} (-1)^l\,\mathrm{Si}(l\pi). \end{aligned}$$

The last equality and (10)imply that for the partial sum in (8) we have $(n > j+1)$

$$\begin{aligned} \int_0^{m_n} |\psi_j| &= (-1)^{n+j+1} (\mathrm{Si}(n-j)\pi - \mathrm{Si}(n+j-1)\pi)/2 + \mathrm{Si}(2j-1)\pi \\ &\qquad + 2\sum_{l=1}^{2j-2} (-1)^{l+1}\,\mathrm{Si}(l\pi) + \sum_{l=n-j}^{n+j-1} (-1)^l\,\mathrm{Si}(l\pi). \end{aligned} \tag{11}$$

For the last sum it holds that

$$\sum_{l=n-j}^{n+j-1} (-1)^l \operatorname{Si}(l\pi) = \sum_{l=1}^{2j} (-1)^{n-j+l-1} \operatorname{Si}(n-j+l-1)\pi$$

$$= (-1)^{n-j} \sum_{l=1}^{j} (\operatorname{Si}(n-j+2l-2)\pi - \operatorname{Si}(n-j+2l-1)\pi).$$

Since $\operatorname{Si}(x) \to \pi/2$ as $x \to \infty$, from (11) after taking the limit as $n \to \infty$ we have

$$\int_0^\infty |\psi_j| = \operatorname{Si}(2j-1)\pi + 2 \sum_{l=1}^{2j-2} (-1)^{l+1} \operatorname{Si}(l\pi). \tag{12}$$

By the definition of the integral sine

$$\operatorname{Si}(2j-1)\pi = \sum_{l=0}^{2j-2} \int_{l\pi}^{(l+1)\pi} \frac{\sin t}{t}\, dt = \sum_{l=0}^{2j-2} (-1)^l \int_0^\pi \frac{\sin u}{u+l\pi}\, du$$

$$= \int_0^\pi \frac{\sin u}{u}\, du + \sum_{l=1}^{j-1} \left(\int_0^\pi \frac{\sin u}{u+2l\pi}\, du - \int_0^\pi \frac{\sin u}{u+(2l-1)\pi}\, du \right).$$

Once more along the same lines we get

$$\sum_{l=1}^{2j-2} (-1)^{l+1} \operatorname{Si}(l\pi) = \sum_{l=1}^{j-1} \int_0^\pi \frac{\sin u}{u+(2l-1)}\, du,$$

and (12) and (5) establish our first assertion.

In view of the inequality

$$\frac{1}{\pi}\left(\frac{1}{2}+\frac{1}{3}+ \cdots + \frac{1}{2j-1}\right) \leqslant \sum_{k=1}^{2j-2} \frac{1}{t+k\pi} \leqslant \frac{1}{\pi}\left(1+\frac{1}{2}+ \cdots + \frac{1}{2j-2}\right),$$

valid for $0 \leqslant t \leqslant \pi$, it follows that

$$\sum_{k=1}^{2j-2} \frac{1}{t+k\pi} = \frac{1}{\pi} \log j + O(1).$$

The second assertion may now be obtained from the first one. This completes the proof.

Next we establish that the generalized Rogosinski means $R_{n,j}$ $(j \in \mathbf{N})$ have the same order of approximation as the classical one $(j=1)$.

LEMMA 1. *Let $S_n f$ be the nth partial sum of the Fourier series of $f \in C_{2\pi}$. Then*

$$R_{n,j}(f, x) = \tfrac{1}{2}\,[S_n(f, x + m_j/n) + S_n(f, x - m_j/n)]. \tag{13}$$

This is an immediate consequence of some trigonometric computation.

Remark. The equality (13) has often been considered as the definition of the Rogosinski means (cf. [1, Chap. VII, Sect. 4]).

Lemma 1 yields the following statement.

PROPOSITION. *Let $E_n(f)$ be the best approximation of $f \in C_{2\pi}$ by trigonometric polynomials of degree not exceeding n, and $\omega_2(f, \delta)$ be the second modulus of continuity. Then*

$$\|f - R_{n,j} f\|_{C_{2\pi}} \leqslant (1 + \|R_{n,j}\|)\, E_n(f) + \frac{1}{2}\,\omega_2\left(f, \frac{m_j}{n}\right).$$

The proof, which is similar to that of Theorem 2.4.8 in [2], is omitted.

Using the Theorem of Jackson and properties of the modulus of continuity, it follows from the proposition that there exists a constant $M_j = O(j^2)$ for which

$$\|f - R_{n,j} f\|_{C_{2\pi}} \leqslant M_j \omega_2\left(f, \frac{1}{n}\right).$$

Thus, for the Rogosinski means $R_{n,j}$ the order of approximation is the same for all $j \in \mathbf{N}$. Moreover by ([2, Problem 1.5.3(iv) and Lemma 1.5.4]), for $f \in W^2$

$$\omega_2\left(f, \frac{1}{n}\right) = O(n^{-2})$$

and better properties of f such as $f \in W^r$ $(r > 2)$ do not imply better approximation. Similar arguments are valid also for arbitrary means for which (4) holds. This is the main reason why we shall consider the quantity (2) only for the classes W^1 and W^2. For more refined Lipschitz classes our method is still not good enough.

3. MEASURE OF APPROXIMATION FOR SOME MEANS OF FOURIER SERIES

We shall consider the measure of approximation (2) for the operators (1) defined by a function φ for which the representation (3) holds. An integral representation of the generalized Rogosinski means is needed.

LEMMA 2. *For all 2π-periodic functions f, Lebesgue integrable over $(-\pi, \pi)$, we have*

$$R_{n,j}(f, x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \psi_j(u) \, f\left(x - \frac{u}{n}\right) du,$$

where the function ψ_j has the representation (7).

The *proof* is similar to the classical ($j = 1$) one (cf. [6, pp. 182–183]).

COROLLARY 1. *Let the operator U_n be defined by the function φ in* (3) *for which*

$$\sum_{j=1}^{\infty} |a_j| \log j < \infty.$$

Then for $f \in C_{2\pi}$ we have

$$U_n(f, x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \psi(u) \, f\left(x - \frac{u}{n}\right) du,$$

where

$$\psi(u) := \sum_{j=1}^{\infty} a_j \psi_j(u) \tag{14}$$

and ψ_j has the representation (7).

Proof. From (4) it follows that

$$U_n(f, x) = \sum_{j=1}^{\infty} a_j R_{n,j}(f, x).$$

Hence, using the Lebesgue dominated convergence theorem, Lemma 2, and Theorem 1, we verify that Corollary 1 is valid.

Now we are ready to prove our main result.

THEOREM 2. *Let the operator U_n be defined by the function φ in* (3) *for which*

$$\sum_{j=1}^{\infty} |a_j| \log j < \infty.$$

If for $\varepsilon > 0$

$$\Phi_1(t) := \int_t^\infty \left(\int_0^1 \varphi(u) \cos ux \, du \right) dx = O(t^{-1-\varepsilon}) \qquad (t \to \infty), \tag{15}$$

then

$$e(W^1, U_n) = \frac{2}{\pi n} \int_0^\infty |\Phi_1(t)| \, dt + O(n^{-1-\varepsilon}) \qquad (n \to \infty).$$

Proof. By Proposition 4.1.1 of [6] we have

$$e(W^1, U_n) = \sup_{f \in W_0^1} |U_n(f, 0)|, \tag{16}$$

where the subclass $W_0^1 \subset W^1$ consists of functions f for which $f(0) = 0$. By Corollary 1 it follows that

$$U_n(f, 0) = \frac{1}{\pi} \int_{-\infty}^\infty \psi(u) \, f\left(-\frac{u}{n}\right) du = \frac{1}{\pi} \int_0^\infty \psi(u) \left(f\left(\frac{u}{n}\right) + f\left(-\frac{u}{n}\right) \right) du \tag{17}$$

by using the evenness of ψ (by (14) and (7)). Since $(f(t) + f(-t))/2 \in W_0^1$ provided $f \in W_0^1$, then

$$e(W^1, U_n) = \frac{2}{\pi} \sup_{f \in W_{0,e}^1} \left| \int_0^\infty \psi(t) \, f\left(\frac{t}{n}\right) dt \right|, \tag{18}$$

where $W_{0,e}^1$ denotes the class of all even functions belonging to W_0^1.

The function ψ in (14) can be written in the form

$$\psi(x) = \int_0^1 \varphi(t) \cos xt \, dt \tag{19}$$

due to (3) and (6). So for Φ_1 in (15) we have

$$\Phi_1(t) = \int_t^\infty \psi(u) \, du, \tag{20}$$

and integration by parts in (18) gives

$$e(W^1, U_n) = \frac{2}{\pi n} \sup_{f \in W_{0,e}^1} \left| \int_0^\infty \Phi_1(t) \, f'\left(\frac{t}{n}\right) dt \right|. \tag{21}$$

Thus, the definition of $W^1_{0,e}$ yields

$$e(W^1, U_n) \leqslant \frac{2}{\pi n} \int_0^\infty |\Phi_1(t)|\, dt, \tag{22}$$

and we will prove that (22) is asymptotically exact.

For the proof we define an even 2π -periodic function f_1 which for $0 \leqslant t \leqslant \pi$ has the form

$$f_1(t) := \int_0^t \operatorname{sgn} \Phi_1(nu)\, du.$$

Therefore, $f_1 \in W^1_{0,e}$ and

$$f_1'\left(\frac{t}{n}\right) = \operatorname{sgn} \Phi_1(t) \qquad \text{for almost every} \quad t \in [0, n\pi]. \tag{23}$$

By (17) and 23) we have

$$\begin{aligned} U_n(f_1, 0) &= \frac{2}{\pi} \int_0^\infty \psi(t)\, f_1\left(\frac{t}{n}\right) dt \\ &= \frac{2}{\pi n} \int_0^\infty \Phi_1(t)\, f_1'\left(\frac{t}{n}\right) dt \\ &= \frac{2}{\pi n} \int_0^\infty |\Phi_1(t)|\, dt + \frac{2}{\pi n} \alpha_n, \end{aligned} \tag{24}$$

where

$$\alpha_n := \int_{n\pi}^\infty \left[f_1'\left(\frac{t}{n}\right) - \operatorname{sgn} \Phi_1(t) \right] \Phi_1(t)\, dt.$$

Since

$$|\alpha_n| \leqslant 2 \int_{n\pi}^\infty |\Phi_1(t)|\, dt,$$

we obtain $\alpha_n = O(n^{-\varepsilon})$ using the assumption (15). Finally, the proof is completed due to (16) and (24).

For the class W^2 the following result is valid.

THEOREM 3. *Let the operator U_n be defined by the function φ in* (3) *for which*

$$\sum_{j=1}^{\infty} |a_j| \log j < \infty.$$

If for $\varepsilon > 0$,

$$\Phi_2(t) := \int_t^{\infty} dv \int_v^{\infty} \left(\int_0^1 \varphi(u) \cos ux \, du \right) dx = O(t^{-1-\varepsilon}) \qquad (t \to \infty), \qquad (25)$$

then we have

$$e(W^2, U_n) = \frac{2}{\pi n^2} \int_0^{\infty} |\Phi_2(t)| \, dt + O(n^{-2-\varepsilon}) \qquad (n \to \infty).$$

Proof. From (19) and (20) it follows by (25) that

$$\Phi_2(t) = \int_t^{\infty} \Phi_1(u) \, du. \qquad (26)$$

For the class W^2 the equality (18) is valid in the form

$$e(W^2, U_n) = \frac{2}{\pi} \sup_{f \in W^2_{0,e}} \left| \int_0^{\infty} \psi(t) f\left(\frac{t}{n}\right) dt \right|,$$

where the subclass $W^2_{0,e} \subset W^2$ consists of even functions f for which $f(0) = 0$. By the definition of W^2 the derivative f' is absolutely continuous. Due to the evenness of f the derivative f' is odd and therefore $f'(0) = 0$. Integrating (18) twice by parts and using (26) we obtain

$$e(W^2, U_n) = \frac{2}{\pi n^2} \sup_{f \in W^2_{0,e}} \left| \int_0^{\infty} \Phi_2(t) f''\left(\frac{t}{n}\right) dt \right|.$$

Hence, by the definition of W^2_0 we have

$$e(W^2, U_n) \leqslant \frac{2}{\pi n^2} \int_0^{\infty} |\Phi_2(t)| \, dt. \qquad (27)$$

Let us now construct a function $f_2 \in W^2_{0,e}$ for which the inequality (27) is asymptotically exact. Let

$$f_2(t) := \int_0^t \left(\int_0^x \operatorname{sgn} \Phi_2(nu) \, du \right) dx \qquad \text{for} \quad 0 \leqslant t \leqslant \pi/2,$$

$$f_2(t) := 2f_2(\pi/2) - f_2(\pi - t) \qquad \text{for} \quad \pi/2 \leqslant t \leqslant \pi$$

with an even and 2π-periodic extension for all $t \in \mathbf{R}$. Then $f_2 \in W^2_{0,e}$ and due to this (17)

$$U_n(f_2, 0) = \frac{2}{\pi} \int_0^\infty \psi(t)\, f_2\left(\frac{t}{n}\right) dt = \frac{2}{\pi n^2} \int_0^\infty \Phi_2(t)\, f_2''\left(\frac{t}{n}\right) dt$$

after integration by parts using (20) and (26). By the definition of f_2 we get

$$f_2''\left(\frac{t}{n}\right) = \operatorname{sgn} \Phi_2(t) \qquad \text{for almost every} \quad t \in [0, n\pi/2]$$

which implies

$$U_n(f_2, 0) = \frac{2}{\pi n^2} \int_0^\infty |\Phi_2(t)|\, dt + \frac{2}{\pi n^2} \beta_n,$$

where

$$\beta_n := \int_{n\pi/2}^\infty \left[f_2''\left(\frac{t}{n}\right) - \operatorname{sgn} \Phi_2(t) \right] \Phi_2(t)\, dt.$$

The proof is completed similarly to that of Theorem 2.

4. APPLICATIONS

We give three examples, the generalized Rogosinski means, the Jackson–de la Vallée Poussin summation method, and the Riesz means, to show how Theorems 2 and 3 apply for these cases.

(a) The *generalized Rogosinski means* are defined by

$$\varphi_j(t) = \cos m_j t \qquad (m_j := (j - 1/2)\pi).$$

The case $j = 1$ has been considered by Dzjadyk *et al.* [3], where numerical estimates for the remainders were also obtained.

The representation (3) is valid trivially. For Φ_1 in (15) we obtain by (6) and (7) that

$$\Phi_1(t) = -\tfrac{1}{2}(\operatorname{si}(t - m_j) + \operatorname{si}(t + m_j)),$$

where the integral sine is defined by

$$\operatorname{si}(t) := -\int_t^\infty \frac{\sin x}{x}\, dx. \tag{28}$$

The first equality in (7) implies

$$\begin{aligned}\Phi_1(t) &= (-1)^j m_j \int_t^\infty \frac{\cos x}{x^2 - m_j^2}\, dx \\ &= (-1)^j m_j \left(2 \int_t^\infty \frac{x \sin x}{(x^2 - m_j^2)^2}\, dx - \frac{\sin t}{t^2 - m_j^2} \right) \\ &= O_j(t^{-2}) \qquad (m_j < t \to \infty).\end{aligned}$$

Therefore, the assumption (15) is valid for $\varepsilon = 1$, and the result will be written as

$$e(W^1, R_{n,j}) = \frac{1}{\pi n} \int_0^\infty |\mathrm{si}(t - m_j) + \mathrm{si}(t + m_j)|\, dt + O_j(n^{-2}) \qquad (n \to \infty)$$

for each fixed $j \in \mathbf{N}$.

For the function Φ_2 in (25) we have by (26) that

$$\begin{aligned}\Phi_2(t) &= -\tfrac{1}{2} \int_t^\infty (\mathrm{si}(u - m_j) + \mathrm{si}(u + m_j))\, du \\ &= \tfrac{1}{2}(t - m_j)\, \mathrm{si}(t - m_j) + \tfrac{1}{2}(t + m_j)\, \mathrm{si}(t + m_j).\end{aligned}$$

Using the definition (28) we get

$$\Phi_2(t) = -\frac{1}{2}(t - m_j) \int_{t - m_j}^\infty \frac{\sin x}{x}\, dx - \frac{1}{2}(t + m_j) \int_{t + m_j}^\infty \frac{\sin x}{x}\, dx.$$

Integrating three times by parts yields

$$\begin{aligned}\Phi_2(t) = {} & (-1)^j \frac{m_j \cos t}{t^2 - m_j^2} + (-1)^j \frac{4 m_j t \sin t}{(t^2 - m_j^2)^2} \\ & + 3(t - m_j) \int_{t - m_j}^\infty \frac{\cos x}{x^4}\, dx + 3(t + m_j) \int_{t + m_j}^\infty \frac{\cos x}{x^4}\, dx.\end{aligned}$$

Now $\Phi_2(t) = O_j(t^{-2})$ $(m_j < t \to \infty)$ and therefore the assumption (25) is valid for $\varepsilon = 1$. Thus, Theorem 3 implies

$$\begin{aligned}e(W^2, R_{n,j}) = {} & \frac{1}{\pi n^2} \int_0^\infty |(t - m_j)\, \mathrm{si}(t - m_j) + (t + m_j)\, \mathrm{si}(t + m_j)|\, dt + O_j(n^{-3}) \\ & (n \to \infty)\end{aligned}$$

for each fixed $j \in \mathbf{N}$.

(b) The *Jackson–de la Vallée Poussin summation method* can be defined ([2, pp. 130–131, 205, 517]) by

$$\varphi(t) := \begin{cases} 1-6t^2+6t^3, & 0 \leqslant t \leqslant 1/2, \\ 2(1-t)^3, & 1/2 \leqslant t \leqslant 1. \end{cases}$$

Obviously φ'' is continuous. Then integrating by parts twice in (19) gives (cf. [2, p. 516])

$$\psi(x) = \frac{96}{x^4} \sin^4 \frac{x}{4}.$$

The function ψ has some good properties. It is nonnegative and for the Fourier coefficients a_j we have by (3) and (19) that

$$a_j = 2\psi(m_j) = O(j^{-4}).$$

Therefore the series $\sum |a_j| \log j$ converges and due to (20)

$$\Phi_1(t) \leqslant 32/t^3.$$

If we denote the Jackson–de la Vallée Poussin summation method by J_n, then using Theorem 2, after some calculations we obtain the equality

$$e(W^1, J_n) = \frac{12 \log 2}{\pi n} + \frac{c_n}{n^3},$$

where $0 \leqslant -c_n \leqslant 32/\pi^3$.

For Φ_2 by (26) we have

$$\Phi_2(t) \leqslant 16/t^2.$$

From Theorem 3 it follows that

$$e(W^2, J_n) = \frac{6}{n^2} + \frac{c_n}{n^3},$$

where $0 \leqslant -c_n \leqslant 64/\pi^2$.

(c) The Riesz means $R_{n,\delta}$ are defined by

$$\varphi(t) = (1-t^2)^\delta \qquad (\delta > 0).$$

The cosine–transform for this φ is (cf. [2, p.516])

$$\psi(x) = c_\delta J_{\delta+1/2}(x)/x^{\delta+1/2},$$

where J_p is the Bessel function of order p and

$$c_\delta := 2^{\delta - 1/2}\Gamma(\delta + 1)\,\pi^{1/2}.$$

This is intended to shorten the notations in the next part.

We need two well known formulae for the Bessel functions

$$J_p(x) = O(x^{-1/2}) \qquad (x \to \infty), \tag{29}$$

$$\int x^{-p+1} J_p(x)\,dx = -x^{-p+1} J_{p-1}(x). \tag{30}$$

In view of (29) we have $\psi(x) = O(x^{-\delta-1})$ $(x \to \infty)$ which gives by (3) and (19) that $a_j = O(j^{-\delta-1})$ $(j \to \infty)$. Therefore, the series $\sum |a_j| \log j$ is convergent for all $\delta > 0$.

To check the validity of the assumptions (15) and (25) we have to integrate by parts using (30). Then by (20) and (29)

$$\Phi_1(t) = c_\delta \int_t^\infty \frac{J_{\delta+1/2}(x)}{x^{\delta+1/2}}\,dx = c_\delta \frac{J_{\delta-1/2}(t)}{t^{\delta+1/2}} - c_\delta \int_t^\infty \frac{J_{\delta-1/2}(x)}{x^{\delta+3/2}}\,dx = O(t^{-\delta-1})$$

which gives the assumption (15). For the last integral we have by (30) and (29)

$$\int_t^\infty \frac{J_{\delta-1/2}(x)}{x^{\delta+3/2}}\,dx = \frac{J_{\delta-3/2}(t)}{t^{\delta+3/2}} - 3\int_t^\infty \frac{J_{\delta-3/2}(x)}{x^{\delta+5/2}}\,dx = O(t^{-\delta-2}).$$

Therefore,

$$\Phi_1(t) = c_\delta \frac{J_{\delta-1/2}(t)}{t^{\delta+1/2}} + O(t^{-\delta-2}) \qquad (t \to \infty)$$

and we have for Φ_2 by (26)

$$\Phi_2(t) = c_\delta \int_t^\infty \frac{J_{\delta-1/2}(x)}{x^{\delta+1/2}}\,dx + O(t^{-\delta-1}). \tag{31}$$

In addition, by (30)

$$\int_t^\infty \frac{J_{\delta-1/2}(x)}{x^{\delta+1/2}}\,dx = \frac{J_{\delta-3/2}(t)}{t^{\delta+1/2}} - 2\int_t^\infty \frac{J_{\delta-3/2}(x)}{x^{\delta+3/2}}\,dx = O(t^{-\delta-1}),$$

and then by (31)

$$\Phi_2(t) = O(t^{-\delta-1}) \qquad (t \to \infty).$$

Thus, all the conditions of Theorems 2 and 3 are fulfilled, the result is as follows:

The Riesz means have the measures of approximation

$$e(W^1, R_{n,\delta}) = \frac{2^{\delta+1/2}\Gamma(\delta+1)}{\pi^{3/2}n} \int_0^\infty \left| \int_t^\infty \frac{J_{\delta+1/2}(x)}{x^{\delta+1/2}}\, dx \right| dt + O(n^{-\delta-1}) \qquad (n \to \infty),$$

$$e(W^2, R_{n,\delta}) = \frac{2^{\delta+1/2}\Gamma(\delta+1)}{\pi^{3/2}n^2} \int_0^\infty t \left| \int_t^\infty \frac{J_{\delta-1/2}(x)}{x^{\delta+3/2}}\, dx \right| dt + O(n^{-\delta-2}) \qquad (n \to \infty)$$

valid for all $\delta > 0$.

Remark. The results fit well with the case $\delta = 1$ considered in [12] as an example.

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